and here the prime q and the exponent β have to be chosen so that p becomes a prime. The error consists in assuming $\lambda = 1$ and hence $q = 2$; that this does not give all the solutions, is evident from the examples:

$$q = 2, \quad \beta = 2, 3, 5 \qquad q = 3, \quad \beta = 2, 3, 5 \qquad q = 5, \quad \beta = 3, 5$$
$$p = 3, 5, 17 \qquad\qquad p = 7, 19, 163 \qquad\qquad p = 101, 2501.$$

214. (April, 1914.) **Proposed by A. J. KEMPNER, University of Illinois.**

Let a be a positive integer $\geqq 2$, and let $T(n)$ denote the number of distinct divisors of the positive integer n, including both 1 and n, so that $T(1) = 1$, $T(2) = 2$, $T(3) = 2$, $T(4) = 3$, $\cdots$. Show that

$$\sum_{n=1}^{n=\infty} T(n)/a^n = \sum_{n=1}^{n=\infty} 1/(a^n - 1).$$

The special case $a = 10$ gives, as is easily seen:

$$9 \sum_{n=1}^{n=\infty} \frac{T(n)}{10^n} = \frac{1}{1} + \frac{1}{11} + \frac{1}{111} + \frac{1}{1111} + \cdots.$$

SOLUTION BY FRANK IRWIN, University of California.

We prove the proposition first for the special case, $a = 10$. We have,

$$1/9 = .11111111\cdots, \qquad 1/99 = .01010101\cdots,$$
$$1/999 = .001001001\cdots, \qquad 1/9999 = .00010001\cdots,$$

and so on.

Let us determine the sum of all the digits in the nth decimal place. Since the kth row of the above array reads
$$1/99\cdots99 = .00\cdots00100\cdots001\cdots,$$

with k 9's on the left and $(k - 1)$ 0's in each recurring period on the right, it follows that we get a 1 in the nth decimal place whenever k is a divisor of n, and otherwise a zero. We have, then, $T(n)$ units in this place, and since a unit there has the value $1/10^n$, their sum is $T(n)/10^n$; and the sum of our series, $1/9 + 1/99 + 1/999 + \cdots$, is equal to

$$\sum_{n=1}^{n=\infty} T(n)/10^n,$$

as was to be proved.

(It is clear that, regarding the array as a double series, we have a right to put the sum by rows equal to the sum by columns.)

In the general case, where we have any a, we need merely suppose the decimals above written in the scale of a, instead of in that of 10,

$$1/(a - 1) = .1111\cdots, \quad 1/(a^2 - 1) = .0101\cdots, \text{ etc.,}$$

and a like argument holds.

QUESTIONS AND DISCUSSIONS.

DISCUSSIONS.

I. RELATING TO NAPIER'S LOGARITHMIC CONCEPT.

By H. S. CARSLAW, University of Sydney, Australia.

In the March number of the MONTHLY, page 71, Professor Cajori takes exception to the following remark, contained in a paper of mine in the *Mathematical Gazette* (Vol. VIII, page 77):

It is sometimes stated that Napier's Logarithms were obtained from the coördination of two definite series, an arithmetical and a geometrical. For instance, Cajori, in his recent paper on the "History of Logarithms," says:

"Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$v, \quad v\left(1 - \frac{1}{v}\right), \quad v\left(1 - \frac{1}{v}\right)^2, \quad \cdots \quad v\left(1 - \frac{1}{v}\right)^n, \quad \cdots,$$

$$0, \qquad 1 \quad , \qquad 2 \qquad , \quad \cdots \qquad n \qquad , \quad \cdots.$$

The numbers in the upper series represent successive values of the *sines*; the numbers in the lower series stand for the corresponding logarithms. Thus $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log [10^7(1 - 10^{-7})^n] = n$, where $n = 0, 1, 2, \cdots$."

This statement is incorrect. In Napier's Tables the logarithm of $(10^7 - 1)$ is not 1. It lies between 1 and 1.000,000,1, and he takes it as the mean between these two numbers, namely 1.000,000,05.

I am sorry that Cajori does not also quote the next sentence in my paper: "If Napier had used these two series in the way named above, his work would have more closely resembled that of Bürgi than actually is the case." These words, and the other references to Bürgi's Tables in my paper, were meant to direct attention to the distinction to which I referred.

Bürgi's work was entitled *Arithmetische und Geometrische Progresstabulen*, and the point I was anxious to make clear was that the fundamental conception in Napier's definition of a logarithm involved far more than the coördination of an arithmetical and a geometrical progression. If Napier's logarithms had been the numbers in the series

$$0, \ 1, \ 2, \ \cdots \quad \text{(A.P.)},$$

and his anti-logarithms (in our use of the term), the numbers

$$10^7, \ 10^7\left(1 - \frac{1}{10^7}\right), \ 10^7\left(1 - \frac{1}{10^7}\right)^2, \ \cdots \quad \text{(G.P.)},$$

then the only difference between his work and that of Bürgi would have been that the latter employed the series

$$10 \times 0, \qquad 10 \times 1, \qquad 10 \times 2, \ \cdots \qquad 10 \times n, \ \cdots \quad \text{(A.P.)},$$

$$10^8, \ 10^8\left(1 + \frac{1}{10^4}\right), \ 10^8\left(1 + \frac{1}{10^4}\right)^2, \ \cdots \ 10^8\left(1 + \frac{1}{10^4}\right)^n, \ \cdots \quad \text{(G.P.)},$$

instead of the A.P. and G.P. which Cajori quotes.

It is, of course, true that geometrical series occur in Napier's work. The first 101 terms of the above series are, in fact, the numbers of his "First Table." And such series were employed in his *computation*, because, with his definition of a logarithm, the logarithms of numbers in geometrical progression have a constant difference.

Cajori argues in his last communication that it is in the *computation*, not in the *definition*, or *theory*, that Napier passes from the coördination of the arithmetical and geometrical progressions. In this I cannot agree with him. The kinematical definition of a logarithm (*cf. Constructio*, § 26) indicates, as I have

said in another place,[1] "that he had already in his mind, though it may be very dimly and vaguely, the principles on which Newton, some 50 years later, built up the differential calculus." And it is one of the most amazing features of Napier's work that his kinematical definition of a logarithm should have suggested itself to him before the invention of the calculus.

It seems to me that Cajori reads into our interpretation of the word logarithm —*the number of the ratios*—that the fundamental conception in Napier's work was the coördination of the arithmetical and geometrical series. My view rather is that Napier started with this coördination—the idea was not new—and very probably invented his word logarithm in connection with it. Later, I believe, he advanced to the much more general principle involved in his definition and treatment.[2] However, as Gibson remarks,[3] "Napier alone knew the derivation of the word, and dogmatism in the matter is out of place."

SYDNEY, AUSTRALIA,
    April, 1916.

COMMENT BY FLORIAN CAJORI, Colorado College.

In the *Mathematical Gazette* of May, 1915, p. 78, Professor Carslaw criticized me for writing Nap. log $(10^7-1)=1$ instead of Nap. log $(10^7-1)=1.00000005$. In my reply[4] I quoted from Napier's *Constructio* to show that Napier himself considered both values as differing "insensibly" from the true value and that my statement was therefore not "incorrect." I made some other observations but made no reference to Bürgian logarithms, since they are irrelevant to the question.

In the present note Professor Carslaw no longer offers any objection to Nap. log $(10^7-1)=1$, but claims that I ignore the kinematical definition of logarithms given by Napier. While in my former reply I laid no emphasis upon that point (it had not been raised in Professor Carslaw's first criticism), I said enough to show that I had it in mind. What else could my reference to Napier's "theory of moving points" mean?

That Professor Carslaw should claim that I ignored Napier's kinematical definition in my *History of the Exponential and Logarithmic Concepts*, published three years ago, is an indication that he has not read with care my account of Napier. I lay very special emphasis upon that definition, where I say:[5]

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[1] *Proc. R. S. New South Wales*, Vol. XLVIII, p. 55, 1914.

[2] In his study of "Logarithms and Computation" in *The Napier Tercentenary Volume* the great authority on this subject—Dr. J. W. L. Glaisher—refers more than once to the fact that Napier's manner of conceiving a logarithm involved quite a new principle. *E. g.*, on p. 69 he says: "I find no difficulty in perceiving that logarithms might have been introduced at that time in such a manner, as we know that Jobst Bürgi did actually conceive antilogarithms, *i. e.*, as a correspondence between $(1.0001)^r$ and $10r$, for integral values of r, with interpolations; but Napier's conception of a logarithm was of a much more subtle kind, and involved the principle of a mathematical function."

[3] Cf. *The Napier Tercentenary Volume*, p. 115.

[4] AMERICAN MATHEMATICAL MONTHLY, vol. 23, p. 72.

[5] *Ibid.*, vol. xx, p. 6.